

Renormalization Group Study of Magnetic Catalysis in the 3d Gross-Neveu Model

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Magnetic catalysis describes the enhancement of symmetry breaking quantum fluctuations in chirally symmetric quantum field theories by the coupling of fermionic degrees of freedom to a magnetic background configuration. We use the functional renormalization group to investigate this phenomenon for interacting Dirac fermions propagating in (2+1)-dimensional space-time, described by the Gross-Neveu model. We identify pointlike operators up to quartic fermionic terms that can be generated in the renormalization group flow by the presence of an external magnetic field. We employ the beta function for the fermionic coupling to quantitatively analyze the field dependence of the induced spectral gap. Within our pointlike truncation, the renormalization group flow provides a simple picture for magnetic catalysis.

I. INTRODUCTION

In the past 20 years it has become evident that in certain planar condensed matter model systems defined on some lattice, the low-energy physics can efficiently be described in terms of Dirac fermions. These “relativistic” degrees of freedom typically arise by linearizing a given dispersion relation around a special set of points in the Brillouin zone, where either a linear band crossing is realized or the single-particle gap function vanishes at isolated points across a given Fermi surface.

The most prominent and well-investigated example is probably the Hubbard model on the honeycomb lattice [1, 2]. This model describing the hopping of fermions on a hexagonal lattice with possibly strong on-site interactions serves as a minimal model for the electronic correlations in monolayer graphene and similar systems. Once the naive continuum limit has been taken, one can expect a faithful representation of the low-energy sector by a quantum field theory for Dirac fermions moving in (2 + 1)-dimensional space-time in the proximity to a second order quantum phase transition. The lattice symmetries of the Hubbard model get replaced by an “emergent” Lorentz symmetry and the free part of the Dirac action is invariant under continuous chiral transformations. Note that for the chiral symmetry to be realized, a reducible four-component representation of the Dirac algebra is necessary, which can be formed from the direct sum of two inequivalent two-component representations. In the case of an irreducible representation, the chiral symmetry effectively corresponds to part of the flavor symmetry. In the following, we will consider the chirally invariant case. This asymptotic chiral symmetry is also realized for example in d -wave superconductors [3, 4] or quantum spin-liquid phases [5, 6]. In both cases, however, a symmetry of the underlying parent state is already assumed to be broken, in contrast to the case of the Hubbard model on the honeycomb lattice. Another situation where no symmetry breaking is necessary are topological insulators and superconductors, which support Dirac fermions as surface states [7, 8].

Once interactions are included in such a low-energy effective quantum field theory, we expect that the chiral

symmetry is spontaneously broken as soon as the associated coupling exceeds a critical value. The fluctuation-driven interactions compatible with the symmetries determine in which channel the formation of a condensate, i.e., a finite expectation value of an order-parameter field, can take place. In this paper, the focus of our investigation is on the Gross-Neveu model with a reducible representation of the Dirac algebra and a purely pointlike scalar-scalar interaction. This theory realizes a system with discrete chiral symmetry. Quantum phase transitions falling into the Gross-Neveu universality class for two fermion flavors $N_f = 2$ are, for example the excitonic [9] and antiferromagnetic instabilities [10] in graphite and graphene, respectively. The universal properties of secondary d - to $d + i$ -wave pairing transitions in nodal d -wave superconductors are also expected to be described by the Gross-Neveu model [3, 4]. Chiral symmetry breaking is known to occur for all flavor numbers N_f [11], whereas in models with continuous chiral symmetry and interactions in a vector-vector channel such as the Thirring model [12], a chiral condensate is stable only up to a critical flavor number.

Remarkably, when interacting planar Dirac fermions are subject to an external perpendicular magnetic field, symmetry breaking accompanied by the generation of a mass gap for the fermionic degrees of freedom occurs for all values of the interaction strength [13]. Indeed, this effect known by the name of magnetic catalysis is not particular to models defined in three space-time dimensions, but has also been investigated in four dimensional quantum field theories [14, 15]. However, what is special about three space-time dimensions is the fact that even in the absence of interactions a finite value of the chiral condensate is generated by applying an external magnetic field [13]. So also for a free system the (continuous) chiral symmetry is spontaneously broken. This phenomenon is sometimes referred to as a ‘quantum anomaly’ [16], since it unexpectedly leads to the generation of a non-vanishing expectation value for a certain operator. For condensed matter model systems, possible applications of this effect have been studied for a variety of low-dimensional physical realizations, ranging from magnetic field induced insulating behavior in semi-metallic structures [17, 18]

to anomalous Hall plateaus in graphene [16, 19–23] and transport properties in d -wave superconductors [24–26]. It is interesting to note that magnetic catalysis can also go along with an induced anomalous magnetic moment of the fermionic excitations [27].

The interplay between strong magnetic fields and chiral-symmetry breaking has also become of topical relevance in particle physics. For instance, non-central heavy-ion collisions go along with extreme magnetic fields [28] that may even interfere with the topological structure of the QCD vacuum [29, 30] or simply take a parametric influence on the chiral phases of QCD [31–37]. Even though magnetic fields in strongly-interacting fermionic systems generically support chiral-condensate formation at least in mean-field-type approximations, further interactions such as gluonic back-reactions may also lead to inverse effects [38].

There exist already a few works in the literature on the large- N_f results for the Gross-Neveu model [39–41]. Beyond leading-order results in a $1/N_f$ -expansion, magnetic catalysis was studied both with truncated Dyson-Schwinger equations (in $d = 3$ [14, 42] and $d = 4$ [14, 43, 44]) and Wilson-style renormalization group (RG) equations (in $d = 4$ [45–47]) in the context of fermionic models with short-ranged interactions and QED. However, a clear RG picture in terms of a fixed-point analysis of the beta function appears to be still missing. Moreover, the inclusion of purely magnetically induced operators in the RG flow and their effect on the running coupling has not been considered so far. In this paper, we provide a clear renormalization mechanism for magnetic catalysis in the Gross-Neveu model. Similar observations are made in an RG study of magnetic catalysis in a QCD low-energy model [48]. Moreover, we identify terms that are compatible with the symmetries of the Gross-Neveu action in the presence of an external magnetic field. In principle, all operators compatible with the symmetries of the initial action can be generated under RG transformations. Therefore, our analysis addresses the question as to whether the operator content of fermionic effective actions is modified in the IR by the very presence of such gauge backgrounds.

The paper is organized as follows. In Sect. II, we define the three-dimensional, chirally symmetric Gross-Neveu model in an external magnetic field. In Sect. III, we collect the transformation rules under both discrete symmetry and chiral transformations, which we will then employ in Sect. IV to study the Gross-Neveu theory space. Put differently, we give the transformation properties of all bilinear operators and list all quartic operators compatible with the Gross-Neveu symmetries. This includes the set of magnetically induced operators (up to a specific mass dimension), the presence of which is completely sustained by the external magnetic field. To keep the presentation self-contained, in Sect. V we recapitulate the functional RG essentials for the one-particle irreducible (1PI) scheme. In Sect. VI we discuss the effect of the magnetic field on the behavior of the beta function for

the fermionic coupling in the pointlike approximation in a qualitative manner. A quantitative discussion of the field dependence of the dynamically generated fermion mass inferred from the beta function is deferred to Sect. VII. We conclude with Sect. VIII where we summarize our findings and give an outlook to our future work. Technical details and the derivation of the beta function are provided in Appendices A–C.

II. GROSS-NEVEU MODEL IN AN EXTERNAL MAGNETIC FIELD

We consider the Gross-Neveu model [49] within functional integral quantization. The microscopic degrees of freedom are represented by Grassmann-valued fields defined over three-dimensional Euclidean space-time. For studying the effects of magnetic catalysis, we minimally couple the Dirac fermions to a background gauge-potential that realizes the external magnetic field. The defining local action to be quantized is given by

$$S[\bar{\psi}_j, \psi_j, \mathcal{A}] = \int_x \mathcal{L}(\bar{\psi}_j, \psi_j, \partial_\mu \bar{\psi}_j, \partial_\mu \psi_j, \mathcal{A}) \quad (1)$$

with the Lagrangian density

$$\mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}_j i \not{D}[\mathcal{A}] \psi_j + \sum_{i,j=1}^{N_f} \bar{\psi}_i \psi_i \frac{\bar{g}}{2N_f} \bar{\psi}_j \psi_j, \quad (2)$$

where $\int_x = \int d^d x$ is a shorthand for the integral over the d -dimensional Euclidean space-time. Here,

$$\not{D}[\mathcal{A}] = \gamma_\mu (\partial_\mu - iq\mathcal{A}_\mu(x)), \quad \mu = 0, 1, 2, \quad (3)$$

denotes the covariant derivative acting on the spinor fields with q the respective charge under the Abelian $U(1)$ gauge group. We consider N_f different flavor species indexed by $j = 1, \dots, N_f$, but keep the flavor number arbitrary. This model enjoys a global $U(N_f)$ flavor symmetry. The canonical mass dimensions of the fields are given by $[\psi] = \frac{d-1}{2}$ and $[\mathcal{A}] = \frac{d-2}{2}$. This renders the gauge charge a dimensionful quantity with $[q] = \frac{4-d}{2}$. As mentioned in Sect. I, we use a reducible four-component representation for the gamma matrices in this work, i.e., $d_\gamma = 4$ where $d_\gamma = \text{tr} \mathbb{1}_4$ denotes the dimension of the representation space of the Dirac algebra and tr is the trace over spinor indices. Our choice for the 4×4 representation of the Dirac algebra is the so-called chiral one. Here, the γ_5 matrix is in block-diagonal form. The matrices are explicitly given by

$$\gamma_0 = \tau_2 \otimes \tau_3, \quad \gamma_1 = \tau_2 \otimes \tau_1, \quad \gamma_2 = \tau_2 \otimes \tau_2 \quad (4)$$

and

$$\gamma_3 = \tau_1 \otimes \tau_0, \quad \gamma_5 = \tau_3 \otimes \tau_0, \quad \gamma_{35} \equiv i\gamma_3\gamma_5 \quad (5)$$

Here, the $\{\tau_i\}$'s denote the Pauli matrices which satisfy $\tau_i \tau_j = \delta_{ij} \tau_0 + i\epsilon_{ijk} \tau_k$, with $i, j, k = 1, 2, 3$ and $\tau_0 = \mathbb{1}_2$

is a 2×2 unit matrix. The gamma matrices satisfy the anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_4, \quad (6)$$

where $\mu, \nu = 0, 1, 2$. The matrices γ_3 and γ_5 anti-commute with all γ_μ , $\mu = 0, 1, 2$. In this representation, γ_2, γ_3 and γ_5 are symmetric and real, while γ_0, γ_1 and γ_{35} are antisymmetric and purely imaginary 4×4 matrices. One easily verifies the following identities

$$\gamma_{35}\gamma_\mu\gamma_\nu = \delta_{\mu\nu}\gamma_{35} + i\epsilon_{\mu\nu\sigma}\gamma_\sigma, \quad (7)$$

$$i\gamma_5\gamma_\mu\gamma_\nu = i\delta_{\mu\nu}\gamma_5 + i\epsilon_{\mu\nu\sigma}\gamma_3\gamma_\sigma, \quad (8)$$

$$i\gamma_3\gamma_\mu\gamma_\nu = i\delta_{\mu\nu}\gamma_3 - i\epsilon_{\mu\nu\sigma}\gamma_5\gamma_\sigma, \quad (9)$$

where the second and third lines follow from the first one by left or right multiplication with an appropriate gamma matrix and making use of the Dirac algebra, and $\epsilon_{\mu\nu\sigma}$ is the completely antisymmetric tensor.

We chose the vector potential in the gauge

$$A(x) = (0, 0, x_1 B)^T. \quad (10)$$

Within the three-dimensional formulation, the magnetic field appears as a pseudo-scalar quantity. This corresponds to the fact that the magnetic flux associated to a magnetic field that is aligned perpendicularly to the (x_1, x_2) -plane can penetrate this plane with two different orientations.

At zero magnetic field, $B \equiv 0$, the model depends on a single parameter [11], namely, the dimensionful coupling constant \bar{g} with mass dimension $2 - d$. In the sense of statistical physics, the coupling appears to correspond to an irrelevant operator within the perturbative Gaussian classification. However, this naive scaling analysis does not yield the correct picture for the infrared behavior of this model, as we will explain in detail in Sect. VI. From the point of view of quantum field theory, the Gross-Neveu model provides an example of a nonperturbatively renormalizable field theory [50], and the $(\bar{\psi}\psi)^2$ coupling becomes RG relevant near a non-Gaussian fixed point [11].

III. GROSS-NEVEU SYMMETRIES

While the construction of local operators that conform to Lorentz symmetry can be ensured by taking products of operators and performing suitable contractions over tensor indices, the transformation properties under discrete space-time symmetries require more work.

In Ref. [12] explicit representations for discrete space-time transformations as realized on 4-component spinor fields over three dimensional Euclidean space were given as follows:

$$\mathcal{C} : C_\xi = \frac{1}{2} [(1 + \xi)\gamma_2\gamma_3 + i(1 - \xi)\gamma_2\gamma_5], \quad (11)$$

$$\mathcal{P} : P_\zeta = \frac{1}{2} [(1 + \zeta)\gamma_1\gamma_3 + i(1 - \zeta)\gamma_1\gamma_5], \quad (12)$$

$$\mathcal{T} : T_\eta = \frac{1}{2} [(1 + \eta)\gamma_1 + i(1 - \eta)\gamma_2\gamma_0], \quad (13)$$

where $\mathcal{C}, \mathcal{P}, \mathcal{T}$ denote charge conjugation, parity inversion and time reversal. To each transformation, there exists an associated unitary 4×4 matrix, which we denote by C_ξ, P_ζ and T_η , respectively. Other possible conventions were given for example in [51]. However, we will stick to the definition as displayed in Eqs. (11), (12) and (13). Concerning parity, it is worthwhile to mention that in three space-time dimensions, parity inversion is properly defined by $(x_0, x_1, x_2) \mapsto (x_0, -x_1, x_2)$, i.e. only one of the spatial components is reversed. This is due to the fact, that in our case, only a single generator for rotations exists. The above definition ensures that parity inversion is not an element of the connected component of rotations containing the identity. As can be seen in Eqs. (11)-(13), there exists an entire family of realizations of discrete transformations depending on pure phase variables ξ, ζ and η with unit modulus. We will simply set $\xi = \zeta = \eta = 1$ in the following and omit the subscripts on the transformation matrices. The realization of space-time transformations on spinor fields reads for charge conjugation

$$\mathcal{C}\psi(x)\mathcal{C}^{-1} = (\bar{\psi}C)^T, \quad \mathcal{C}\bar{\psi}(x)\mathcal{C}^{-1} = -(C^\dagger\psi(x))^T, \quad (14)$$

parity

$$\mathcal{P}\psi(x)\mathcal{P}^{-1} = P\psi(\tilde{x}), \quad \mathcal{P}\bar{\psi}(\tilde{x})\mathcal{P}^{-1} = \bar{\psi}(\tilde{x})P^\dagger, \quad (15)$$

and time-reversal

$$\mathcal{T}\psi(x)\mathcal{T}^{-1} = T\psi(\hat{x}), \quad \mathcal{T}\bar{\psi}(x)\mathcal{T}^{-1} = \bar{\psi}(\hat{x})T^\dagger. \quad (16)$$

Here we follow the notation as given in [51], and define

$$\tilde{x} = (x_0, -x_1, x_2)^T, \quad \hat{x} = (-x_0, x_1, x_2)^T. \quad (17)$$

Note that by virtue of $\mathcal{T}i\mathcal{T}^{-1} = -i$, time-reversal is an anti-unitary transformation. We will consider a theory to be symmetric under \mathcal{C} -, \mathcal{P} - and \mathcal{T} -transformations, if its Lagrangian density obeys

$$\mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L}(x), \quad (18)$$

$$\mathcal{P}\mathcal{L}(x)\mathcal{P}^{-1} = \mathcal{L}(\tilde{x}), \quad (19)$$

$$\mathcal{T}\mathcal{L}(x)\mathcal{T}^{-1} = \mathcal{L}(\hat{x}), \quad (20)$$

such that a simultaneous transformation acting on fields *and* coordinates leaves \mathcal{L} invariant as a function of space-time coordinates.

The Gross-Neveu model as defined in Eq. (2) is symmetric under the *discrete* chiral transformation

$$\psi \mapsto \gamma_5\psi, \quad \bar{\psi} \mapsto -\bar{\psi}\gamma_5. \quad (21)$$

Since the composite operator $\bar{\psi}\psi$ transforms into $-\bar{\psi}\psi$ under this discrete chiral transformation, a finite expectation value $\langle\bar{\psi}\psi\rangle \neq 0$ in a given quantum state signals the breakdown of chiral symmetry. In the following, we denote the discrete symmetry group by $\mathbb{Z}_2^5 = \{\mathbb{1}, \gamma_5\}$. However, there exists also a continuous Abelian chiral symmetry, generated by γ_{35} :

$$\psi \mapsto e^{i\varphi\gamma_{35}}\psi, \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\varphi\gamma_{35}}. \quad (22)$$

It is easy to see that the element $e^{i\frac{\pi}{2}\gamma_{35}} \in U^{35}(1)$ combined with the non-trivial \mathbb{Z}_2^3 transformation

$$\psi \mapsto \gamma_3 \psi, \quad \bar{\psi} \mapsto -\bar{\psi} \gamma_3 \quad (23)$$

leads us back to the transformation Eq. (21). In this respect, the discrete \mathbb{Z}_2^3 does not yield a new symmetry of the theory and we will henceforth choose \mathbb{Z}_2^5 and $U^{35}(1)$ to define the chiral symmetries of the three dimensional Gross-Neveu model. It is perhaps worth mentioning that for the free theory, i.e. $\bar{g} = 0$, the symmetry transformations act independently on all flavor species j , $j = 1, \dots, N_f$. For finite couplings, however, all flavors are subject to a simultaneous chiral \mathbb{Z}_2^5 transformation, as we expect from the symmetry-breaking pattern induced by the interaction term.

To conclude this section, we summarize the properties of electromagnetic quantities under discrete space-time transformations. The gauge potential \mathcal{A} transforms as

$$\mathcal{C}\mathcal{A}_\mu(x)\mathcal{C}^{-1} = -\mathcal{A}_\mu(x), \quad (24)$$

$$\mathcal{P}\mathcal{A}_\mu(x)\mathcal{P}^{-1} = \tilde{\mathcal{A}}_\mu(\tilde{x}), \quad (25)$$

$$\mathcal{T}\mathcal{A}_\mu(x)\mathcal{T}^{-1} = -\hat{\mathcal{A}}_\mu(\hat{x}), \quad (26)$$

where $\tilde{\mathcal{A}} = (\mathcal{A}_0, -\mathcal{A}_1, \mathcal{A}_2)^T$ and $\hat{\mathcal{A}} = (-\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)^T$. The field-strength tensor $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ accordingly obeys

$$\mathcal{C}\mathcal{F}_{\mu\nu}(x)\mathcal{C}^{-1} = -\mathcal{F}_{\mu\nu}(x), \quad (27)$$

$$\mathcal{P}\mathcal{F}_{\mu\nu}(x)\mathcal{P}^{-1} = \tilde{\mathcal{F}}_{\mu\nu}(\tilde{x}), \quad (28)$$

$$\mathcal{T}\mathcal{F}_{\mu\nu}(x)\mathcal{T}^{-1} = -\hat{\mathcal{F}}_{\mu\nu}(\hat{x}). \quad (29)$$

Here, by $\tilde{\mathcal{F}}_{\mu\nu}$ and $\hat{\mathcal{F}}_{\mu\nu}$ we denote matrices which result from plugging Eqs. (25) and (26) into the definition of the field-strength tensor. The so-called dual ‘field-strength’ $F_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\sigma}\mathcal{F}_{\nu\sigma}$ becomes a pseudo-vector quantity in three dimensions. Thus, it behaves as

$$\mathcal{C}F_\mu(x)\mathcal{C}^{-1} = -F_\mu(x), \quad (30)$$

$$\mathcal{P}F_\mu(x)\mathcal{P}^{-1} = -\tilde{F}_\mu(\tilde{x}), \quad (31)$$

$$\mathcal{T}F_\mu(x)\mathcal{T}^{-1} = \hat{F}_\mu(\hat{x}). \quad (32)$$

The simplest Lorentz, gauge and \mathcal{C} -, \mathcal{P} -, \mathcal{T} -invariants that can be built from the field strength and its dual are

$$\mathcal{F}_{\mu\nu}^2 \quad \text{and} \quad F_\mu^2. \quad (33)$$

Since $F_\mu^2 = \frac{1}{2}\mathcal{F}_{\mu\nu}^2$ there is only one linearly independent invariant.

IV. CLASSIFICATION OF COMPATIBLE OPERATORS

Having collected the prerequisites to study all operators that are in principle compatible symmetry-wise with the Lagrangian density Eq. (2), we will give an exhaustive classification on the level of fermionic bilinears in Sect. IV A and then proceed to quartic fermionic terms including purely magnetically induced operators in Sect. IV B.

A. Bilinear Operators

Fermionic bilinears are the building blocks of an action for fermionic degrees of freedom. Terms entering the quadratic part of the action define the inverse bare propagator of the theory. In our case, we need to contract both spinor and Lorentz indices to form an appropriate scalar quantity. But, due to the presence of the gauge background, we can as well use the gauge-invariant field-strength tensor and its dual to build Lorentz-invariant fermion bilinears. We consider only those bilinears which are Lorentz symmetric as a building block of the effective action. By use of Eqs. (14)–(16) and Eqs. (21) and (22) we obtain the behavior of a given bilinear under discrete space-time and chiral transformations. The results are collected in Tables I, II, and III. For turning operators with vector or tensor structure into Lorentz-invariant scalars, we contract as indicated above with F_μ or $\mathcal{F}_{\mu\nu}$. By virtue of the canonical mass dimension of the fermion fields $[\psi] = \frac{d-1}{2}$, the mass dimension of a (non-derivative) fermion bilinear is $[\bar{\psi}\Gamma\psi] = d-1$ with $\Gamma_S \in \{\mathbb{1}, \gamma_3, \gamma_5, \gamma_{35}\}$ (scalar/pseudo-scalar), $\Gamma_V \in \{\gamma_\mu, \gamma_3\gamma_\mu, \gamma_5\gamma_\mu, \gamma_{35}\gamma_\mu\}$ (vector/axial vector), or $\Gamma_T \in \{\sigma_{\mu\nu}, \gamma_3\sigma_{\mu\nu}, \gamma_5\sigma_{\mu\nu}, \gamma_{35}\sigma_{\mu\nu}\}$ (tensor/pseudotensor). Here,

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu], \quad \mu, \nu = 0, 1, 2, \quad (34)$$

is the set of generators for (Euclidean) Lorentz transformations. For the contracted terms we obtain $[qF_\mu(\bar{\psi}\Gamma_\mu\psi)] = d+1$ and $[q\mathcal{F}_{\mu\nu}(\bar{\psi}\Gamma_{\mu\nu}\psi)] = d+1$, since $[F_\mu] = [\mathcal{F}_{\mu\nu}] = \frac{d}{2}$ and $[q] = \frac{4-d}{2}$. Taking into account that the space-time integral in a local action contributes $[\int_x] = -d$ to the total mass dimension of a given local operator, these operators naively correspond to irrelevant directions in theory space. The charge q needs to be included in this consideration, since we are interested in magnetically *induced* phenomena. We could also perform two contractions of qF_μ with an appropriate tensor structure. This would inevitably increase the mass dimension by 2 and render this term even more power-counting irrelevant compared to mass terms (with mass dimension -1) and kinetic operators with mass dimension 0. However, from Tables I–III, we see that none of these bilinear operators containing the magnetic field are compatible with the symmetries of the Gross-Neveu model. They cannot be generated during the RG flow in a continuous fashion. Combinations like $\mathcal{F}_{\mu\nu}(\bar{\psi}\sigma_{\mu\nu}\psi)$ and $F_\mu(\bar{\psi}\gamma_{35}\gamma_\mu\psi)$ conform to space-time symmetries, but violate the chiral symmetries of the Gross-Neveu action. Chirally invariant combinations containing two F_μ fields are not even symmetric with respect to \mathcal{C} , \mathcal{P} and \mathcal{T} .

To conclude, we would like to comment that each F_μ contraction with an appropriate vector bilinear can be rewritten as a contraction of $\mathcal{F}_{\mu\nu}$ with a tensor bilinear

by means of Eqs. (7)-(9):

$$F_\mu(\bar{\psi}\gamma_\mu\psi) = -\frac{1}{2}\mathcal{F}_{\mu\nu}(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi), \quad (35)$$

$$F_\mu(\bar{\psi}\gamma_3\gamma_\mu\psi) = -\frac{i}{2}\mathcal{F}_{\mu\nu}(\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi), \quad (36)$$

$$F_\mu(\bar{\psi}\gamma_5\gamma_\mu\psi) = +\frac{i}{2}\mathcal{F}_{\mu\nu}(\bar{\psi}\gamma_3\sigma_{\mu\nu}\psi), \quad (37)$$

$$F_\mu(\bar{\psi}\gamma_{35}\gamma_\mu\psi) = -\frac{1}{2}\mathcal{F}_{\mu\nu}(\bar{\psi}\sigma_{\mu\nu}\psi). \quad (38)$$

B. Quartic Operators

Quartic fermionic terms describe the two-body interaction processes in our theory. We first ask for terms

$$\sum_{i,j=1}^{N_f} (\bar{\psi}_i\Gamma_X\psi_i)(\bar{\psi}_j\Gamma_Y\psi_j) \quad (39)$$

which obey the symmetries of the original Gross-Neveu action and have canonical mass dimension $[(\bar{\psi}_i\Gamma_X\psi_i)(\bar{\psi}_j\Gamma_Y\psi_j)] = 2d - 2$. In a next step, we analyze contributions with mass dimension $2d$, i.e., quartic terms that include a contraction with a mass dimension 2 object. In principle, one could also move to operators with higher mass dimension. But with increasing mass dimension our naive expectation is that these operators become increasingly irrelevant for the IR behavior of the theory, even near a non-Gaussian fixed point.

Note that terms as captured by Eq. (39) are composed of flavor singlets. Invariant sums of bilinears with off-diagonal flavor structure can be brought into this singlet-singlet form by an appropriate Fierz transformation. The result is in general a sum over several singlet-singlet contributions with different Lorentz structure.

Since a discrete chiral symmetry is less restrictive than continuous chiral symmetry, the number of allowed quartic terms appears to be quite large. However, the requirement for invariance under the *continuous* $U^{35}(1)$ symmetry remedies the situation somewhat. The allowed terms in the absence of a magnetic field are exhausted by $(\bar{\psi}\psi)^2$, $(\bar{\psi}\gamma_\mu\psi)^2$, $(\bar{\psi}\sigma_{\mu\nu}\psi)^2$ and $(\bar{\psi}\gamma_{35}\psi)^2$, $(\bar{\psi}\gamma_{35}\gamma_\mu\psi)^2$, $(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)^2$ as well as the invariant combinations $(\bar{\psi}\gamma_3\psi)^2 + (\bar{\psi}\gamma_5\psi)^2$, $(\bar{\psi}\gamma_3\gamma_\mu\psi)^2 + (\bar{\psi}\gamma_5\gamma_\mu\psi)^2$ where we have omitted all flavor indices. Actually $(\bar{\psi}\gamma_{35}\gamma_\mu\psi)^2$ and $(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)^2$ can be written as

$$(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)^2 = \frac{1}{2}(\bar{\psi}\gamma_\mu\psi)^2, \quad (40)$$

$$(\bar{\psi}\gamma_{35}\gamma_\mu\psi)^2 = \frac{1}{2}(\bar{\psi}\sigma_{\mu\nu}\psi)^2, \quad (41)$$

since they are simply linear combinations of the Dirac algebra basis elements. The remaining six quartic terms are mutually independent. They parametrize the ‘Gross-Neveu theory space’ without an external magnetic field in the pointlike limit. It is interesting to note that an RG flow calculation in the pointlike limit does actually

not make use of this full theory space, but only reproduces the Gross-Neveu coupling in the derivative expansion. This might be accidental or could point to a further hidden symmetry.

We now consider quartic fermionic terms that can be built by contracting once with $q\mathcal{F}_{\mu\nu}$ or qF_μ . Due to the identities Eqs. (35)-(38), it is sufficient to consider contractions with qF_μ only. From Tables I-III we conclude, that the following terms are allowed

$$(\bar{\psi}\psi)(\bar{\psi}\gamma_{35}\gamma_\mu F_\mu\psi) = -\frac{1}{2}(\bar{\psi}\psi)(\bar{\psi}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}\psi), \quad (42)$$

$$(\bar{\psi}\gamma_{35}\psi)(\bar{\psi}\gamma_\mu F_\mu\psi), \quad (43)$$

and

$$(\bar{\psi}\gamma_3\psi)(\bar{\psi}\gamma_5\gamma_\mu F_\mu\psi) + (\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_3\gamma_\mu F_\mu\psi). \quad (44)$$

Of course, a $U^{35}(1)$ invariant operator with fully contracted Lorentz indices squared is also compatible with the Gross-Neveu symmetries, but has mass dimension larger or equal to $2d + 2$. One could also consider functions of the invariant $(qF_\mu)^2$ times a bilinear squared. This, however, corresponds to operators with arbitrarily high mass dimension. In Appendix B we show that starting from the naive Gross-Neveu action Eq. (1), a term of the type Eq. (42) is indeed generated in an infinitesimal RG step. In phases with broken chiral symmetry, such an operator may give rise to an anomalous magnetic moment, as observed in [27].

V. FUNCTIONAL RG EQUATION

In this section we briefly summarize the essentials of the (functional) renormalization group. In the space of couplings g_i that parametrize a given action, the RG provides a vector field β , summarizing the RG β functions for these couplings $(\beta)_i = \beta_{g_i}(g_1, g_2, \dots) \equiv \partial_t g_i$. As the full content of a quantum theory can be specified in terms of generating functionals for correlation functions or vertices, we can more generally study the RG behavior of a generating functional. Introducing an IR-regulated effective average action Γ_k by inserting a scale-dependent regulator into the Gaussian measure of the functional integral representing the generating functional of vacuum correlators, $Z_k[J]$, the RG flow of this action is determined by the Wetterich equation [52]

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{Str} \left\{ \left[\Gamma_k^{(2)}[\Phi] + R_k \right]^{-1} (\partial_t R_k) \right\}, \quad \partial_t = k \frac{d}{dk}. \quad (45)$$

More precisely, $\Gamma_k[\Phi]$ is the modified Legendre transform

$$\Gamma_k[\Phi] = (J, \Phi) - W_k[J] - \frac{1}{2}(\Phi, R_k \Phi) \quad (46)$$

of the scale-dependent generating functional

$$W_k[J] = \log Z_k[J] \quad (47)$$

	\mathcal{C}	\mathcal{P}	\mathcal{T}	\mathbf{Z}_2^5	$\mathbf{U}^{35}(1)$
$(\bar{\psi}\psi)(x)$	$(\bar{\psi}\psi)(x)$	$(\bar{\psi}\psi)(\tilde{x})$	$(\bar{\psi}\psi)(\hat{x})$	$-(\bar{\psi}\psi)(x)$	$(\bar{\psi}\psi)(x)$
$(\bar{\psi}\gamma_3\psi)(x)$	$-(\bar{\psi}\gamma_3\psi)(x)$	$-(\bar{\psi}\gamma_3\psi)(\tilde{x})$	$-(\bar{\psi}\gamma_3\psi)(\hat{x})$	$(\bar{\psi}\gamma_3\psi)(x)$	$(\bar{\psi}\gamma_3 e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_5\psi)(x)$	$(\bar{\psi}\gamma_5\psi)(x)$	$(\bar{\psi}\gamma_5\psi)(\tilde{x})$	$-(\bar{\psi}\gamma_5\psi)(\hat{x})$	$-(\bar{\psi}\gamma_5\psi)(x)$	$(\bar{\psi}\gamma_5 e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_{35}\psi)(x)$	$(\bar{\psi}\gamma_{35}\psi)(x)$	$-(\bar{\psi}\gamma_{35}\psi)(\tilde{x})$	$-(\bar{\psi}\gamma_{35}\psi)(\hat{x})$	$(\bar{\psi}\gamma_{35}\psi)(x)$	$(\bar{\psi}\gamma_{35}\psi)(x)$

Table I. Classification of scalar/pseudo-scalar fermion bilinears $(\bar{\psi}\Gamma_S\psi)(x)$ according to their behavior under discrete space-time and chiral transformations.

	\mathcal{C}	\mathcal{P}	\mathcal{T}	\mathbf{Z}_2^5	$\mathbf{U}^{35}(1)$
$(\bar{\psi}\gamma_\mu\psi)(x)$	$-(\bar{\psi}\gamma_\mu\psi)(x)$	$(\bar{\psi}\tilde{\gamma}_\mu\psi)(\tilde{x})$	$-(\bar{\psi}\hat{\gamma}_\mu\psi)(\hat{x})$	$(\bar{\psi}\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_\mu\psi)(x)$
$(\bar{\psi}\gamma_3\gamma_\mu\psi)(x)$	$-(\bar{\psi}\gamma_3\gamma_\mu\psi)(x)$	$-(\bar{\psi}\gamma_3\tilde{\gamma}_\mu\psi)(\tilde{x})$	$(\bar{\psi}\gamma_3\hat{\gamma}_\mu\psi)(\hat{x})$	$-(\bar{\psi}\gamma_3\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_3\gamma_\mu e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_5\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_5\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_5\tilde{\gamma}_\mu\psi)(\tilde{x})$	$(\bar{\psi}\gamma_5\hat{\gamma}_\mu\psi)(\hat{x})$	$(\bar{\psi}\gamma_5\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_5\gamma_\mu e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_{35}\gamma_\mu\psi)(x)$	$-(\bar{\psi}\gamma_{35}\gamma_\mu\psi)(x)$	$-(\bar{\psi}\gamma_{35}\tilde{\gamma}_\mu\psi)(\tilde{x})$	$(\bar{\psi}\gamma_{35}\hat{\gamma}_\mu\psi)(\hat{x})$	$-(\bar{\psi}\gamma_{35}\gamma_\mu\psi)(x)$	$(\bar{\psi}\gamma_{35}\gamma_\mu\psi)(x)$

Table II. Classification of vector/axial-vector fermion bilinears $(\bar{\psi}\Gamma_V\psi)(x)$ according to their behavior under discrete space-time and chiral transformations. Here, we have defined $\tilde{\gamma} = (\gamma_0, -\gamma_1, \gamma_2)^T$ and $\hat{\gamma} = (-\gamma_0, \gamma_1, \gamma_2)^T$. The bilinear $\bar{\psi}\gamma_{35}\gamma_\mu\psi$ can be shown to be related to $\bar{\psi}\sigma_{\mu\nu}\psi$, cf. Eq. (7).

	\mathcal{C}	\mathcal{P}	\mathcal{T}	\mathbf{Z}_2^5	$\mathbf{U}^{35}(1)$
$(\bar{\psi}\sigma_{\mu\nu}\psi)(x)$	$-(\bar{\psi}\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\tilde{\sigma}_{\mu\nu}\psi)(\tilde{x})$	$(\bar{\psi}\hat{\sigma}_{\mu\nu}\psi)(\hat{x})$	$-(\bar{\psi}\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\sigma_{\mu\nu}\psi)(x)$
$(\bar{\psi}\gamma_3\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\gamma_3\sigma_{\mu\nu}\psi)(x)$	$-(\bar{\psi}\gamma_3\tilde{\sigma}_{\mu\nu}\psi)(\tilde{x})$	$(\bar{\psi}\gamma_3\hat{\sigma}_{\mu\nu}\psi)(\hat{x})$	$(\bar{\psi}\gamma_3\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\gamma_3\sigma_{\mu\nu} e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi)(x)$	$-(\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\gamma_5\tilde{\sigma}_{\mu\nu}\psi)(\tilde{x})$	$(\bar{\psi}\gamma_5\hat{\sigma}_{\mu\nu}\psi)(\hat{x})$	$-(\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\gamma_5\sigma_{\mu\nu} e^{2i\varphi\gamma_3}\psi)(x)$
$(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)(x)$	$-(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)(x)$	$-(\bar{\psi}\gamma_{35}\tilde{\sigma}_{\mu\nu}\psi)(\tilde{x})$	$(\bar{\psi}\gamma_{35}\hat{\sigma}_{\mu\nu}\psi)(\hat{x})$	$(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)(x)$	$(\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi)(x)$

Table III. Classification of tensor/pseudo-tensor fermion bilinears $(\bar{\psi}\Gamma_T\psi)(x)$ according to their behavior under discrete space-time and chiral transformations. The matrices $\tilde{\sigma}_{\mu\nu}$ and $\hat{\sigma}_{\mu\nu}$ carry the same sign structure as $\tilde{F}_{\mu\nu}$ and $\hat{F}_{\mu\nu}$ in Eq. (28) and (29), respectively. The bilinear $\bar{\psi}\gamma_{35}\sigma_{\mu\nu}\psi$ can be shown to be equivalent to $\bar{\psi}\gamma_\mu\psi$, cf. Eq. (9). It also holds that $\bar{\psi}\gamma_{3/5}\sigma_{\mu\nu}\psi \sim \bar{\psi}\gamma_{5/3}\gamma_\mu\psi$.

for connected correlators and $\Gamma_k^{(2)}$ is the second functional derivative with respect to the field Φ , representing an appropriate collection of field variables for all bosonic or fermionic degrees of freedom. The brackets (\cdot, \cdot) are shorthand for suitable contractions of generalized indices. The function R_k denotes a momentum-dependent regulator that suppresses IR modes below a momentum scale k . STr denotes the so-called super-trace operation, which simply takes into account the minus sign from the loop contributions from the fermionic sector. The solution to the Wetterich equation provides for an RG trajectory in the space of all action functionals, also known as *theory space*. The trajectory interpolates between the bare action S_Λ to be quantized $\Gamma_{k \rightarrow \Lambda} \rightarrow S_\Lambda$ and the full quantum effective action $\Gamma = \Gamma_{k \rightarrow 0}$, being the generating functional of 1PI correlation functions; for reviews with a focus on fermionic systems, see [53–56] and particularly [57]. Parametrizing the effective average action Γ_k by a set of generalized dimensionless couplings g_i , the Wetterich equation provides us with the corresponding RG flow $\partial_t g_i = \beta_{g_i}(g_1, g_2, \dots)$. A fixed point $g_{i,*}$ is defined by

$$\beta_i(g_{1,*}, g_{2,*}, \dots) = 0 \quad \forall i. \quad (48)$$

In general, this map of a quantum field theory to a flow in coupling space allows to extract universal physical information from the fixed points and from the associated manifolds controlling the flow toward the infrared.

VI. MAGNETIC BETA FUNCTION

In the following, we will elaborate on how the phenomenon of magnetic catalysis manifests itself in the IR behavior of the beta function for the coupling \bar{g} . The beta function for vanishing gauge background was derived previously within the formalism of the functional renormalization group [11]. Explicitly, it is given by

$$\beta_g \equiv \partial_t g = (d-2)g - 4 \left(\frac{N_f d_\gamma - 2}{N_f} \right) v_d l_1^F(0) g^2 \quad (49)$$

with the dimensionless coupling

$$g = k^{d-2} \bar{g}. \quad (50)$$

In Eq. (49), $d_\gamma = \text{tr} \mathbb{1}_4 = 4$ for the present reducible fermion representation, and $v_d^{-1} = 2^{d+1} \pi^{d/2} \Gamma(d/2)$. For the definition of the threshold function $l_1^F(\omega)$ parametrizing the RG-scheme dependence, see Appendix C. Aside from the Gaussian fixed point of a noninteracting theory,

$$\partial_t g \propto (d-2)g - \left(\frac{4N_f-2}{N_f}\right) \tilde{\partial}_t \left[\text{diagram of a fermion loop with two external gauge lines and two red dots labeled } g \right]$$

$$\text{diagram of a fermion line with a cross} = \text{diagram of a fermion line} + \text{diagram of a fermion line with a wavy line and a cross} + \text{diagram of a fermion line with two wavy lines and two crosses} + \dots$$

Figure 1. The upper part depicts a diagrammatic representation of the beta function for the dimensionless coupling g (red dot). The regularized fermionic loop contains the full B dependence as given by the fermionic propagator coupling to the external gauge field \mathcal{A} to all orders as shown in the lower part. The coupling to \mathcal{A} is indicated by a cross. The $\tilde{\partial}_t$ derivative acts on the regulator-dependent part of the propagators and generates diagrams with regulator insertions (for details, see Appendix B).

this beta function has also a non-Gaussian, i.e., strong-coupling fixed point. At this non-Gaussian fixed point the scaling dimension is altered from $\Theta_{\text{Gauss}}(\bar{g}) < 0$ at the Gaussian one to $\Theta_{\text{nonGauss}}(\bar{g}) > 0$, cf. [11]. So in contrast to a naive scaling analysis, the coupling actually corresponds to a relevant direction in theory space at this non-trivial zero of the beta function. This fixed-point structure is related to the known nonperturbative renormalizability of the Gross-Neveu model. The renormalization program was performed within a $1/N_f$ -expansion [50], and it was argued that the theory is renormalizable to all orders in this expansion. With the help of the renormalization group, the theory was shown to be asymptotically safe, i.e., the RG flow contains trajectories that eventually hit the non-trivial fixed point with finite dimensionless couplings if the scale k (and implicitly the UV cutoff Λ) is sent to infinity [11]. In this model, the non-trivial fixed point also acts as a separatrix for different regimes of the flow. These two different regimes translate into different phases realized in the respective ground states of the quantum field theory [52, 58]. For values $g < g_*$, the flow is toward a non-interacting theory, while for $g > g_*$ the coupling diverges at some finite IR scale k_c , indicating that the discrete chiral symmetry is spontaneously broken. Thus, g_* can be interpreted as the critical value of g corresponding to a chiral quantum phase transition. Whereas g_* is generally non-universal, the universal (scheme-independent) critical exponents of this quantum phase transition can be determined quantitatively with very good agreement among the various methods [11, 59–61]. The order parameter for chiral symmetry breaking is the so-called chiral condensate, i.e., the expectation value of the operator $\sum_{i=1}^{N_f} \bar{\psi}_i \psi_i$. It can be convenient to introduce an order-parameter field σ into the functional integral by means of a Hubbard-Stratonovich transformation. However, for giving a simple renormalization group picture of magnetic catalysis and also for avoiding technicalities, we stay within a pointlike, purely fermionic description. For

finite flavor number, we expect our arguments to be reliable until the onset of chiral symmetry breaking, where a proper description of the built up of a finite expectation value of the order parameter and the associated collective fluctuations need to be taken into account.

To capture the effects of a magnetic field on the RG flow, we make the following ansatz for the effective average action:

$$\Gamma_k[\bar{\psi}_j, \psi_j, \mathcal{A}] = \int_x \left\{ \sum_{j=1}^{N_f} \bar{\psi}_j i \not{D}[\mathcal{A}] \psi_j + \sum_{i,j=1}^{N_f} \bar{\psi}_i \psi_i \frac{\bar{g}}{2N_f} \bar{\psi}_j \psi_j \right\}, \quad (51)$$

where the coupling \bar{g} is now to be understood as a function of the RG scale k . Inserting this ansatz into the flow equation, we can project onto the flow of the dimensionless coupling g by taking appropriate functional derivatives with respect to the spinor fields. The presence of the magnetic field requires a slightly different approach in the derivation of the beta function, which is usually performed for the corresponding momentum-space quantities and fields. One problem is that the vector potential $\mathcal{A}(x)$ entering the quadratic part of the action renders the fermionic propagator a translationally non-invariant function of the space-time coordinates. Taking gauge holonomy factors appropriately into account, it still becomes possible to study the ansatz in Eq. (51), which essentially constitutes the zeroth order in a derivative expansion. This implies that the derivative structure contained in the effective average action is the same as in the classical action. In Appendix B we demonstrate that Eq. (51) leads to a consistent flow equation within such a derivative expansion. There we also show that the scale dependence of a wave function renormalization entering the action in the form $Z_\psi \sum_{j=1}^{N_f} \bar{\psi}_j i \not{D}[\mathcal{A}] \psi_j$ in the present truncation is trivial, i.e., $Z_\psi \equiv 1$ like in standard flows for pointlike fermionic theories with chiral symmetry [12, 57].

The beta function $\beta_g \equiv \partial_t g$ is derived in Appendix B. It incorporates the coupling of the fermions to the external field to all orders in qB , see Fig. 1, and reads using a Callan-Symanzik regulator

$$\partial_t g = (d-2)g - \frac{1}{16\pi} \left(\frac{N_f d_\gamma - 2}{N_f} \right) g^2 \left[\sqrt{\frac{2}{b}} \zeta \left(\frac{3}{2}, \frac{1}{2b} \right) - 2b \right]. \quad (52)$$

Here, we have defined the dimensionless magnetic field parameter

$$b = \frac{qB}{k^2}. \quad (53)$$

In the limit $b \rightarrow 0$, the correct results for vanishing field are recovered, cf. Eq. (49) and Appendix C. The magnetic field defines a new scale, the so-called magnetic length $l = 1/\sqrt{qB}$. The inverse magnetic length thus describes the typical energy and momentum scales associated with fermions moving in a magnetic field. For

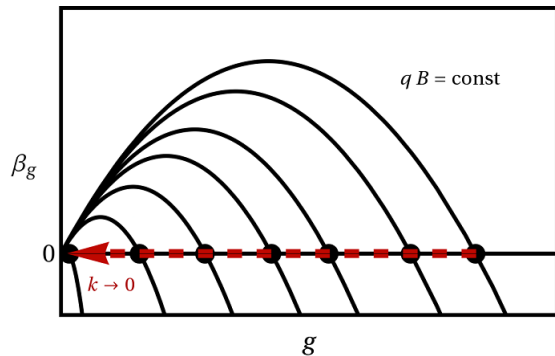


Figure 2. Plot of the beta function β_g (cf. Eq. (52)) of the coupling g in the presence of an external magnetic field B in arbitrary units. As the RG scale k tends to zero for fixed B , the quadratic part of the beta function dominates over the linear part due to dimensional flow. The strong-coupling fixed point (black dots) is pushed toward the Gaussian fixed-point as the scale moves from the high UV to the IR regime (indicated by dashed red arrow). This leads to a divergence in g at a finite scale associated with chiral symmetry breaking, for arbitrary values of the initial coupling at the UV scale Λ . From right to left b takes the values 0.1, 1, 2, 3, 5, 10, 100 and 1000.

strong magnetic fields, the fermionic fluctuations that experience Landau level quantization are primarily driven by the lowest Landau level [$n = 0$, $\tau = -1$ (cf. Appendix A)]. A Landau level can roughly be attributed an extent of l in all spatial directions orthogonal to the magnetic field. The present quantum field theory in $d = 3$ is thus dimensionally reduced via the external field to essentially a quantum mechanical problem ($d \rightarrow d - 2 = 1$).

In the regime $l^{-1} \ll k$, the high-momentum quantum fluctuations on the scale k are not affected by the long-wavelength physics associated with the low-momentum scale l^{-1} . From the initial UV scale $k = \Lambda$ down to the magnetic scale $k = l^{-1}$, we expect the beta function to be very similar to its zero field counterpart. However, as we evolve the system toward the IR by sending the RG scale k to zero, the IR fluctuations in the regime $l^{-1} \gg k$ are dominated by magnetic-field effects.

This behavior is depicted in Fig. 2. At high scales ($b \gg 1$), the beta function shows the typical competition between a positive linear and negative quadratic contribution [cf. Eq. (49)]. Its shape is unaffected by the presence of an external magnetic field. Upon lowering k , the non-Gaussian fixed point starts moving toward the Gaussian fixed point. In the deep IR limit, the non-Gaussian fixed point asymptotically merges with the Gaussian one. However, only the negative branch of the beta function survives in this limit, rendering g a relevant perturbation, completely oblivious of the initial UV value of the coupling. In this way, the divergence of g at a finite scale is catalyzed by the external field for arbitrary values of g .

A complementary approach to this phenomenon is the

so-called “quantum anomaly” [16] for the Lagrangian

$$\mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}_j i \not{D}[\mathcal{A}] \psi_j. \quad (54)$$

It can be shown that the $U(2N_f)$ symmetry of this noninteracting system minimally coupled to a perpendicular magnetic field is *spontaneously* broken down to $U(N_f) \times U(N_f)$ for $B \neq 0$. One finds [13, 63–65]

$$\sum_{j=1}^{N_f} \langle \bar{\psi}_j \psi_j \rangle = -N_f \frac{qB}{2\pi}. \quad (55)$$

However, for vanishing interaction ($\bar{g} = 0$), this expectation value can not couple back into the bilinear part of the theory. So, although the chiral symmetry is broken and a finite chiral condensate is dynamically generated, it does not act as a mass term for the fermions and the spectrum of the theory remains unchanged.

VII. RG ANALYSIS OF MAGNETICALLY INDUCED SPECTRAL GAP

The phenomenon of spontaneous chiral symmetry breaking is accompanied by the dynamical generation of a mass \bar{m}_f for the fermionic degrees of freedom which drive the symmetry breaking. In a partially bosonized formulation, which we already alluded to earlier, the condensation of an order-parameter field σ introduces a mass term into the free part of the Dirac Lagrangian. The onset of symmetry breaking driven by quantum fluctuations typically occurs at some critical scale k_c . We expect that $\bar{m}_f \sim k_c$. The collective fluctuations of the fermionic system encoded in fluctuations of the σ field about its expectation value will typically diminish this value. In cases where a continuous symmetry is broken, the accompanying Nambu-Goldstone modes have the tendency to restore the order, and the expectation value of the order-parameter field can be pushed to smaller values.

The beta function Eq. (52) can now be used to analyze the field dependence of the dynamically induced fermion mass. For this, we use the identification $\bar{m}_f = k_c$, ignoring possible $\mathcal{O}(1)$ factors arising from fluctuations in the symmetry-broken regime. This approximation is equivalent to a gap-equation approach [62]. The critical scale k_c can be identified with the scale where the flow of g diverges. Therefore, it is convenient to reformulate the beta function for g as a beta function for the inverse coupling $1/g$. From $\beta_{1/g}$, we can move to the actual trajectory $1/g(k)$ by solving the flow equation with a given initial condition $g(\Lambda) = g_0$ and record whenever $1/g(k_c) = 0$.

The result of this analysis is shown in Fig. 3. The initial or bare value g_0 was chosen from three different regimes. The flow equation was evaluated for flavor numbers ranging from $N_f = 1$ to $N_f = 100$, which at this level of the truncation is indiscernible from the large- N_f limit. For clarity, only the curves for $N_f = 1$ and $N_f = 100$ are

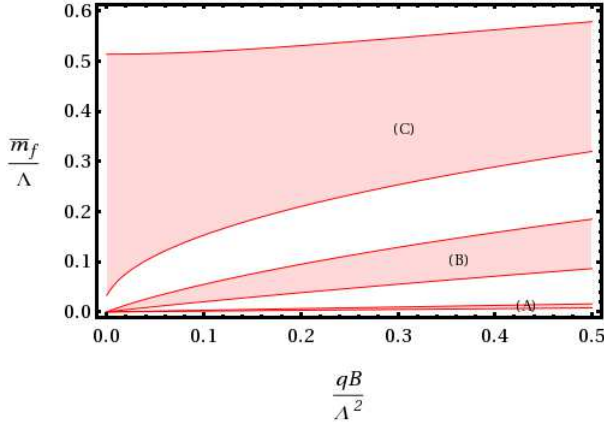


Figure 3. B dependence of the dynamically generated fermion mass \bar{m}_f as determined from the critical scale k_c . The lowest boundary of each band corresponds to $N_f = 1$ and the upper boundary to $N_f = 100$. The curves for intermediate N_f can be computed as easily, but are not shown for clarity. Band (A) is the weak coupling $g_0 \ll g_*$, band (B) the intermediate but sub-critical coupling $g_0 \lesssim g_*$, and (C) the strong coupling regime, $g_0 \gtrsim g_*$.

shown. The curves for $1 < N_f < 100$ fall into the shaded regions and do not cross in the B -interval we scanned. The precise values for g_0 and Λ are listed in Table IV. In the weak coupling regime $g_0 \ll g_*$ we find $\bar{m}_f \propto B$. For intermediate, but sub-critical values of the initial coupling $g_0 \lesssim g_*$ deviations from the linear behavior show up. In a sense, in this regime the generation of a fermion mass is due to the interplay of the ‘quantum anomaly’ of the free system for $B \neq 0$ mentioned in Sect. VI and the four-point vertex g (in the pointlike approximation). While the anomaly generates a finite chiral condensate, the four-point vertex allows for coupling this condensate back into the single particle spectrum, such that a mass gap opens up. In the strong coupling regime $g \gtrsim g_*$ the fermion mass is generated already for $B = 0$ in the usual chiral quantum phase transition from a massless phase to a phase with massive Dirac fermions controlled by the beta function for vanishing field.

Regime	$g_0^{(\text{RG})}$	$\Lambda^{(\text{RG})}$	$g_0^{(\text{DS})}$	$\Lambda^{(\text{DS})}$
(A) $g_0 \ll g_*$	0.20	100 a.u.	0.206	100 a.u.
(B) $g_0 \lesssim g_*$	2.00	100 a.u.	2.76	100 a.u.
(C) $g_0 \gtrsim g_*$	6.50	100 a.u.	65.00	100 a.u.

Table IV. Initial or bare value of the interaction g_0 and cutoff scale Λ in the large- N_f RG and Dyson-Schwinger gap equation. The scheme dependent critical values are $g_*^{(\text{RG})} = \pi$ and $g_*^{(\text{DS})} \simeq 5.39$.

To assess the validity of the pointlike approximation of the four point vertex and the naive identification of the dynamically generated fermion mass \bar{m}_f with the critical scale k_c we use in this work, we compared the large- N_f limit of our flow equation with the large- N_f gap equation.

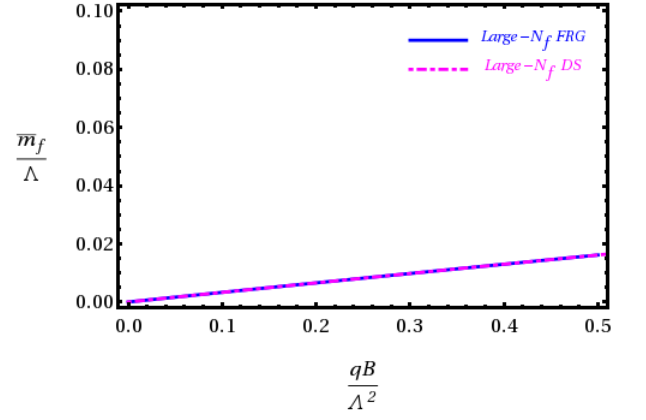


Figure 4. Quantitative comparison of the B dependence of the dynamically generated fermion mass from the large- N_f flow equation (solid blue) and the large- N_f gap equation in the weak-coupling regime. For cutoff Λ and bare value of g_0 , see Table IV.

This equation was derived in e.g. [13] and can be written as

$$1 = \frac{\bar{g}}{2\pi^{\frac{3}{2}}} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t^{\frac{3}{2}}} e^{-\bar{m}_f^2 t} (qBt) \coth(qBt). \quad (56)$$

The integral is regularized in the proper-time formulation and can be performed by analytic continuation, such that a closed but cutoff-dependent expression is obtained (see [13]). The results are shown in Figs. 4-6. The values of the cutoff scale Λ and the bare interaction g_0 used in the two different schemes (RG and Dyson-Schwinger gap equation) are given in Table IV. In order to compare these two approaches, the values of Λ and g_0 were chosen, such that the dynamically generated fermion mass \bar{m}_f obtained from the RG flow and the self-consistent solution to Eq. (56) coincide for $qB/\Lambda^2 = 0.1$. Since the Gross-Neveu model (or its corresponding universality class) only has a single parameter (which is true beyond the pointlike approximation), this fixing prescription is sufficient.

We find excellent agreement over a wide range of qB . Deviations occur due to cutoff effects when $qB/\Lambda^2 \sim 0.35$, as expected for scheme dependencies. Unfortunately, this quantitative check cannot be carried over to finite N_f in a fully controlled manner. Especially the identification $\bar{m}_f = k_c$ cannot be expected to hold any longer, since collective excitations – which are completely damped in the large- N_f limit – will modify the value of \bar{m}_f beyond the critical scale k_c that for finite flavor numbers signals only the onset of chiral symmetry breaking. We also expect the strength of the transition to be weakened upon inclusion of order-parameter fluctuations, as has been recently observed in 4d quark-meson models [66].

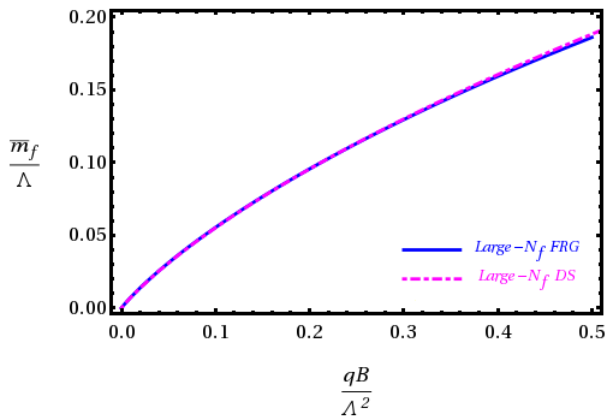


Figure 5. Quantitative comparison of the B dependence of the dynamically generated fermion mass from the large- N_f flow equation (solid blue) and the large- N_f gap equation (dashed-dotted, magenta) in the intermediate to strong coupling regime. The coupling is still sub-critical. For cutoff Λ and bare value of g_0 , see Table IV.

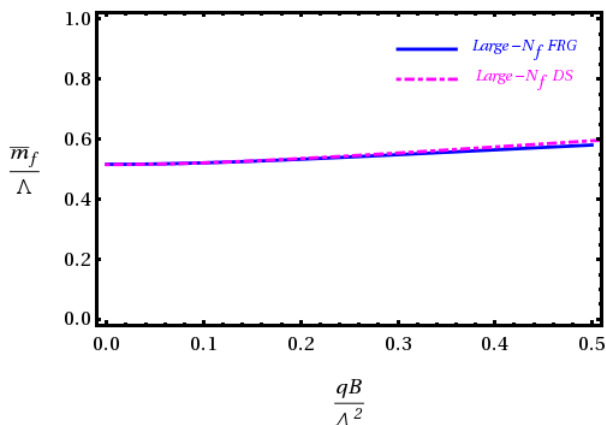


Figure 6. Quantitative comparison of the B dependence of the dynamically generated fermion mass from the large- N_f flow equation (solid, blue) and the large- N_f gap equation (dashed-dotted, magenta) in the strong coupling regime. For cutoff Λ and bare value of g_0 , see Table IV.

VIII. CONCLUSIONS AND OUTLOOK

In this paper, we have studied the phenomenon of magnetic catalysis within the three-dimensional Gross-Neveu model. Minimally coupling the Dirac fermions to an external magnetic field catalyzes chiral symmetry breaking for arbitrary bare values of the fermionic interaction. Even in the free system, chiral symmetry breaking occurs and can be attributed to a form of ‘quantum anomaly’. However, only in the interacting case does the symmetry breaking lead to a mass generation for the fermionic degrees of freedom. Using the functional RG flow equation for the generating functional of 1PI vertices, we have obtained the beta function for the coupling parametrizing the fermionic interaction in a pointlike approxima-

tion. An analysis of the IR behavior of the beta function yielded a clear picture of magnetic catalysis in the language of the renormalization group. The strong-coupling fixed point is pushed toward the Gaussian fixed point. But in the deep IR limit, we find that only the negative branch of the beta function remains, and subsequently the coupling always flows into the strong coupling regime and eventually signals the breakdown of chiral symmetry in a divergence at a finite critical scale.

This fixed-point picture of the approach to criticality in fact occurs in many fermionic systems [57]. For instance, also a gauge-field background giving rise to a non-vanishing Polyakov loop pushes the strong-coupling fixed point toward the Gaussian fixed point [67]. The unique feature of a magnetic background is the occurrence and subsequent IR dominance of the lowest Landau level acting as a symmetry-breaking catalyzer [13, 14].

By a symmetry analysis we provide insight into the structure of the Gross-Neveu theory space and more importantly how additional magnetically induced operators might affect chiral symmetry breaking in external fields. We find that a fermionic tensor-scalar contribution is generated for finite flavor number, when starting from the naive Gross-Neveu action with only scalar-scalar interactions. It is subject to current investigation to study quantitatively the impact of these induced operators on the value of the chiral condensate and the dynamically generated fermion mass in chirally symmetric quantum field theories. The occurrence of these magnetically-induced operators is not particular to the Gross-Neveu model but certainly a generic feature of any fermionic system. In fact, this is yet another example for the proliferation of operators in the presence of further Lorentz tensors as is familiar from the classification of operators, e.g., at finite temperature [57, 68]. It should be emphasized that mean-field studies typically ignore contribution of such terms.

Within our truncation, there is no critical magnetic field, i.e., symmetry breaking occurs for all B . In principle, however, once further interaction channels have opened up, there might be the possibility to compensate the ‘driving force’ from the scalar-scalar channel, such that for certain regimes, a critical field B_c might be necessary to enter the broken phase. This would also imply that the interacting theory is not connected to the free theory in an analytic way as our present analysis suggests, and new scaling laws could be observed.

Also, we neglected the back-reaction of collective neutral excitations onto the fermionic system. For small flavor numbers, we expect sizable quantitative modifications from the fluctuations of the order-parameter field. For infinite flavor numbers, we recover the exact results of the large- N_f theory. The derivation of a consistent set of renormalization group equations for the case of charged fermionic and neutral collective excitations is a work in progress and will be presented elsewhere.

Concerning condensed matter applications, precise quantitative control over the field dependence of the

fermionic single-particle gap might provide understanding of the interplay between magnetic catalysis and interaction-driven anomalous Hall plateaus in quantum Hall measurements.

ACKNOWLEDGMENTS

The authors thank J. Braun, L. Janssen and J.M. Pawłowski for valuable discussions and collaboration on related topics and acknowledge support by the DFG under grants Gi 328/5-1 (Heisenberg program), GRK1523 and FOR 723.

Appendix A: Spectrum of Dirac Operator in an External Magnetic Field

Here we briefly summarize essentials of the spectrum of the square of the operator $i\mathcal{D}[\mathcal{A}]$, such that the trace appearing in the fermion loop (cf. Fig. 1) can be conveniently expressed in the corresponding basis. With the conventions in Sec. II, the squared Dirac operator can be written as

$$\begin{aligned} (i\mathcal{D}[\mathcal{A}])^2 &= -\left(D[\mathcal{A}]^2 \mathbb{1}_4 - \frac{q}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}\right) \\ &= -\left(D[\mathcal{A}]^2 \tau_0 \otimes \tau_0 + qB \tau_0 \otimes \tau_3\right), \end{aligned} \quad (\text{A1})$$

where $D[\mathcal{A}]^2$ is the covariant Laplacian. The spectrum of the squared Dirac operator is thus found to be

$$\{p_0^2 + \epsilon_n^2 + \tau qB \mid p_0 \in \mathbb{R}, n \in \mathbb{N}, \tau = \pm 1\}. \quad (\text{A2})$$

Here, $\epsilon_n^2 = |qB|(2n+1)$ corresponds to Landau level energies and n is the associated Landau level index. Accounting for the density of states $\frac{qB}{2\pi}$ for each Landau level, the trace operation can be decomposed as

$$\text{Tr}(\cdot) = \Omega \sum_{n \in \mathbb{N}} \left(\frac{qB}{2\pi}\right) \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \text{tr}(\cdot), \quad (\text{A3})$$

where Ω denotes space-time volume and tr is the trace over spinor indices.

Appendix B: Derivation of the beta Function

It is convenient to derive the flow equations in position space rather than momentum space due to the presence of the vector potential $\mathcal{A}(x)$. Strictly speaking, the propagator is in general no longer a translationally invariant function of space-time coordinates. It can, however, be decomposed into a holonomy factor times a translationally invariant factor [65], allowing in principle for a momentum-space formulation. Still, a formulation in position space is more direct in our purely pointlike approx-

imation. The flow equation Eq. (45) can be expanded as

$$\begin{aligned} \partial_t \Gamma_k &= \frac{1}{2} \text{STr} \left\{ \left[\Gamma_k^{(2)} + R_k \right]^{-1} (\partial_t R_k) \right\} \\ &= -\frac{1}{2} \text{Tr} \tilde{\partial}_t \ln G_k^{-1} - \frac{1}{2} \text{Tr} \tilde{\partial}_t \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} (G_k \tilde{\Gamma}_k^{(2)})^l, \end{aligned} \quad (\text{B1})$$

where the $\tilde{\partial}_t$ derivative only acts on the regulator dependent part. Here we have decomposed the fluctuation kernel $\Gamma_k^{(2)} = \bar{\Gamma}_k^{(2)} + \tilde{\Gamma}_k^{(2)}$, where $\bar{\Gamma}_k^{(2)}$ is the field-independent part and $\tilde{\Gamma}_k^{(2)}$ denotes the field-dependent vertex part. We also defined the scale-dependent regularized propagator

$$G_k \equiv \left[\Gamma_k^{(2)} + R_k \right]^{-1}. \quad (\text{B2})$$

We choose a chirally invariant regulator

$$R_k = Z_\psi \left[i\mathcal{D}[\mathcal{A}] r \left(\frac{(i\mathcal{D}[\mathcal{A}])^2}{k^2} \right) \right], \quad (\text{B3})$$

with an as yet unspecified shape function $r(x)$. The fluctuation kernel can conveniently be derived in Nambu representation for the spinor fields (see e.g. [69]). We thus obtain

$$G_k = \begin{pmatrix} 0 & G_k^+ \\ G_k^- & 0 \end{pmatrix} \quad (\text{B4})$$

with $G_k^+ = [Z_\psi(i\mathcal{D}[\mathcal{A}]) + R_k]^{-1}$, $G_k^- = [G_k^+]^T$ and

$$\tilde{\Gamma}_{k,ij}^{(2)} = \begin{pmatrix} \bar{H}_{ij} & -F_{ij}^T \\ F_{ij} & H_{ij} \end{pmatrix}. \quad (\text{B5})$$

While we have omitted flavor indices for the propagator part, which is diagonal in flavor space, the flavor structure of the vertex part is non-trivial. The matrix blocks are given by

$$\bar{H}_{ij} = -\frac{\bar{g}}{N_f} \bar{\psi}_i^T \bar{\psi}_j, \quad (\text{B6})$$

$$-F_{ij}^T = \frac{\bar{g}}{N_f} \left\{ \bar{\psi}_i^T \psi_j^T - \sum_{l=1}^{N_f} (\bar{\psi}_l \psi_l) \delta_{ij} \mathbb{1}_4 \right\}, \quad (\text{B7})$$

$$F_{ij}^T = \frac{\bar{g}}{N_f} \left\{ \psi_i \bar{\psi}_j + \sum_{l=1}^{N_f} (\bar{\psi}_l \psi_l) \delta_{ij} \mathbb{1}_4 \right\}, \quad (\text{B8})$$

$$H_{ij} = -\frac{\bar{g}}{N_f} \psi_i \psi_j^T. \quad (\text{B9})$$

We define the position space representation of the operator $i\mathcal{D}[\mathcal{A}]$ as

$$\langle x | i\mathcal{D}[\mathcal{A}] | x' \rangle \equiv i\mathcal{D}[\mathcal{A}](x, x') = \gamma_\mu (i\partial_\mu^x + \mathcal{A}_\mu(x)) \delta(x - x'). \quad (\text{B10})$$

Accordingly, we proceed for the regulator part R_k inserted into the Gaussian measure of the generating functional of Green's functions:

$$\langle x | R_k | x' \rangle \equiv R_k(x, x') = Z_\psi \left[i\mathcal{D}[\mathcal{A}] r \left(\frac{(i\mathcal{D}[\mathcal{A}])^2}{k^2} \right) \right](x, x'). \quad (\text{B11})$$

In this notation, the regularized propagator is given by

$$G_k^+(x, x') = Z_\psi^{-1} \left[i\mathcal{D}[\mathcal{A}] \left(1 + r \left(\frac{(i\mathcal{D}[\mathcal{A}])^2}{k^2} \right) \right) \right]^{-1} (x, x'). \quad (\text{B12})$$

However, the wave function renormalization is $Z_\psi = 1$ in this pointlike truncation. This can be seen from the tadpole diagram giving rise to self-energy corrections. The

regularized tadpole diagram is proportional to

$$\int_x \tilde{\partial}_t \text{tr} G^+(x, x). \quad (\text{B13})$$

Since it can be shown that the above mentioned holonomy factor becomes trivial for $x' = x$, the remaining contribution to the trace comes from the translation invariant part and thus the tadpole carries no net momentum at all, like in the $B = 0$ case.

The vertex part of the fluctuation matrix becomes in position space representation

$$\bar{H}_{ij}(x, x') = -\frac{\bar{g}}{N_f} \bar{\psi}_i^T(x) \bar{\psi}_j(x) \delta(x + x'), \quad (\text{B14})$$

$$-F_{ij}^T(x, x') = \frac{\bar{g}}{N_f} \left\{ \bar{\psi}_i^T(x) \psi_j^T(x) - \sum_{l=1}^{N_f} (\bar{\psi}_l \psi_l)(x) \delta_{ij} \mathbb{1}_4 \right\} \delta(x - x'), \quad (\text{B15})$$

$$F_{ij}^T(x, x') = \frac{\bar{g}}{N_f} \left\{ \psi_i(x) \bar{\psi}_j(x) + \sum_{l=1}^{N_f} (\bar{\psi}_l \psi_l)(x) \delta_{ij} \mathbb{1}_4 \right\} \delta(x - x'), \quad (\text{B16})$$

$$H_{ij}(x, x') = -\frac{\bar{g}}{N_f} \psi_i(x) \psi_j^T(x) \delta(x + x'). \quad (\text{B17})$$

The projection onto the beta function of the dimensionless coupling $g = k^{d-2} \bar{g}$ is facilitated by collecting all contributions on the right-hand side of Eq. (B1), which are quartic in the fermionic fields. We find two contributions, $\frac{1}{2} \text{Tr} \tilde{\partial}_t \{G_k^+ F G_k^+ F\}$ and $\frac{1}{2} \text{Tr} \tilde{\partial}_t \{G_k^+ H G_k^- \bar{H}\}$. These sub-

sequently will be evaluated for spatially constant spinor fields. Using the identity

$$\tilde{\partial}_t G_k^+ = -G_k^+ \partial_t R_k G_k^+, \quad (\text{B18})$$

and after some algebra, we arrive at

$$\begin{aligned} \frac{1}{2} \text{Tr} \tilde{\partial}_t \{G_k^+ F G_k^+ F + G_k^+ H G_k^- \bar{H}\} &= \frac{2}{N_f} \bar{g}^2 \sum_{i,j=1}^{N_f} \int_x \bar{\psi}_i \left(\left[i\mathcal{D}[\mathcal{A}] \left(1 + r \left(\frac{(i\mathcal{D}[\mathcal{A}])^2}{k^2} \right) \right) \right]^{-3} \partial_t R_k \right) (x, x) \psi_i (\bar{\psi}_j \psi_j) \\ &\quad - \bar{g}^2 \text{tr} \int_x \left(\left[i\mathcal{D}[\mathcal{A}] \left(1 + r \left(\frac{(i\mathcal{D}[\mathcal{A}])^2}{k^2} \right) \right) \right]^{-3} \partial_t R_k \right) (x, x) \sum_{i,j=1}^{N_f} (\bar{\psi}_i \psi_i) (\bar{\psi}_j \psi_j), \end{aligned} \quad (\text{B19})$$

where tr denotes the trace over spinor indices. The first term on the right-hand side in the first line of Eq. (B19) contains not only a contribution to the flow of g , but also a term $\sim (\bar{\psi}_i \sigma_{\mu\nu} \mathcal{F}_{\mu\nu} \psi_i) (\bar{\psi}_j \psi_j)$ is generated in an infinitesimal RG step. Nevertheless, in the large- N_f limit, this contribution drops out from the fermionic loop. It is also important to note that the spinors $\bar{\psi}_i$ and ψ_i are to be contracted with the term in round brackets. To see this more clearly, it is suitable to employ a Laplace

transformation,

$$f(x) = \int_0^\infty ds \tilde{f}(s) e^{-xs}, \quad (\text{B20})$$

with $x = (i\mathcal{D}[\mathcal{A}])^2/k^2$, in order to handle the appearing traces. Since the spectrum of $(i\mathcal{D}[\mathcal{A}])^2$ is known (cf. Eq. (A2)), we perform the trace with help of Eq. (A3) and obtain

$$\text{Tr}' e^{-(i\mathcal{D}[\mathcal{A}])^2/k^2 s} = \Omega \left(\frac{qB}{2\pi} \right) \left[\frac{k}{4\sqrt{\pi s} \sinh \left(\frac{qBs}{k^2} \right)} \right] \left[\cosh \left(\frac{qBs}{k^2} \right) \tau_0 \otimes \tau_0 + \sinh \left(\frac{qBs}{k^2} \right) \tau_0 \otimes \tau_3 \right], \quad (\text{B21})$$

where the prime indicates that the trace over spinor indices is not included. The last term in Eq. (B21) makes the generation of a magnetically induced operator explicit. However, neglecting this contribution, the flow equation for g can be derived in closed form by specifying the shape function $r(x)$ as

$$r(x) = \sqrt{\frac{x+1}{x}} - 1, \quad (\text{B22})$$

i.e., by using the Callan-Symanzik regulator. The Laplace transform is known in this case and reads

$$\tilde{f}(s) = -\frac{1}{2}e^{-s}s. \quad (\text{B23})$$

Upon performing the Laplace integral, we finally arrive at the rather simple result

$$\partial_t g = (d-2)g - \frac{1}{16\pi} \left(\frac{N_f d_\gamma - 2}{N_f} \right) g^2 \left[\sqrt{\frac{2}{b}} \zeta \left(\frac{3}{2}, \frac{1}{2b} \right) - 2b \right], \quad (\text{B24})$$

where $b \equiv \frac{qB}{k^2}$ denotes the dimensionless external field and $\zeta(x, y)$ is the Hurwitz zeta function.

Appendix C: Threshold Functions

The regulator dependence of the flow equation is encoded in dimensionless threshold functions which arise from 1PI diagrams describing fermionic quantum fluctuations. In this work we used a Callan-Symanzik regulator. In the case of vanishing magnetic field, the beta function for the coupling g is given by

$$\partial_t g = (d-2)g - 4 \left(\frac{N_f d_\gamma - 2}{N_f} \right) v_d l_1^F(0) g^2, \quad (\text{C1})$$

with the threshold function

$$l_1^F(\omega) = \tilde{\partial}_t \int_0^\infty dy y^{\frac{d}{2}-1} \frac{1}{(1+r(y))^2 + \omega}, \quad (\text{C2})$$

see e.g. [12, 53] for a derivation. For the Callan-Symanzik regulator, the threshold function evaluates to $l_1^F(0) = \frac{\pi}{2}$. We checked that as $b \rightarrow 0$ the beta function Eq. (52) has the correct limit and the results for vanishing field are recovered. This is easily seen by noting that

$$\lim_{b \rightarrow 0} \left[\sqrt{\frac{2}{b}} \zeta \left(\frac{3}{2}, \frac{1}{2b} \right) - 2b \right] = 4. \quad (\text{C3})$$

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